

NOETHER'S PROBLEM FOR \widehat{S}_4 AND \widehat{S}_5

Ming-chang Kang^{a,1} and Jian Zhou^{b,1,2}

^aDepartment of Mathematics and Taida Institute of Mathematical Sciences, National Taiwan University, Taipei

^bSchool of Mathematical Sciences, Peking University, Beijing

Abstract. Let k be a field, G be a finite group and $k(x_g : g \in G)$ be the rational function field over k , on which G acts by k -automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Noether's problem asks whether the fixed subfield $k(G) := k(x_g : g \in G)^G$ is k -rational, i.e. purely transcendental over k . If \widehat{S}_n is the double cover of the symmetric group S_n , in which the liftings of transpositions and products of disjoint transpositions are of order 4, Serre shows that $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not \mathbb{Q} -rational. We will prove that, if k is a field such that $\text{char } k \neq 2, 3$, and $k(\zeta_8)$ is a cyclic extension of k , then $k(\widehat{S}_4)$ is k -rational. If it is assumed furthermore that $\text{char } k = 0$, then $k(\widehat{S}_5)$ is also k -rational.

§1. Introduction

Let k be a field, and L be a finitely generated field extension of k . L is called k -rational (or rational over k) if L is purely transcendental over k , i.e. L is isomorphic to some rational function field over k . L is called stably k -rational if $L(y_1, \dots, y_m)$ is k -rational for some y_1, \dots, y_m which are algebraically independent over k . L is called k -unirational if L is k -isomorphic to a subfield of some k -rational field extension of k . It is easy to see that “ k -rational” \Rightarrow “stably k -rational” \Rightarrow “ k -unirational”.

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E-mail addresses: kang@math.ntu.edu.tw, zhjn@math.pku.edu.cn.

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A notion of retract rationality was introduced by Saltman (see [Sa; Ka2]). It is known that, if k is an infinite field, then “stably k -rational” \Rightarrow “retract k -rational” \Rightarrow “ k -unirational”.

Let k be a field and G be a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k -automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Denote by $k(G)$ the fixed subfield, i.e. $k(G) = k(x_g : g \in G)^G$. Noether’s problem asks, under what situation, the field $k(G)$ is k -rational.

Noether’s problem is related to the inverse Galois problem and the existence of generic G -Galois extensions over k . For the details, see Swan’s survey paper [Sw]. The purpose of this paper is to study Noether’s problem for some double covers of the symmetric group S_n .

It is known that, when $n \geq 4$, there are four different double covers of S_n , i.e. groups G satisfying the short exact sequence $1 \rightarrow C_2 \rightarrow G \rightarrow S_n \rightarrow 1$ (see, for example, [Se, page 653]).

Definition 1.1 ([GMS, p.58, 90; HH, p.18; Kar, p.177-181]) Let $C_2 = \{\pm 1\}$ be the cyclic group of order 2. When $n \geq 4$, the group \widehat{S}_n is the unique central extension of S_n by C_2 , i.e. $1 \rightarrow C_2 \rightarrow \widehat{S}_n \rightarrow S_n \rightarrow 1$, satisfying the condition that the transpositions and the product of two disjoint transpositions in S_n lift to elements of order 4 in \widehat{S}_n . On the other hand, the group \widetilde{S}_n is the central extension $1 \rightarrow C_2 \rightarrow \widetilde{S}_n \rightarrow S_n \rightarrow 1$ such that a transposition in S_n lifts to an element of order 2 of \widetilde{S}_n , but a product of two disjoint transpositions in S_n lifts to an element of order 4.

Note that we follow the notation of \widehat{S}_n and \widetilde{S}_n adopted by Serre in [GMS], which are different from those in [HH].

Using cohomological invariants and trace forms over \mathbb{Q} , Serre was able to prove the following theorem.

Theorem 1.2 (Serre [GMS, p.90]) *Both $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not retract \mathbb{Q} -rational. In particular, they are not \mathbb{Q} -rational.*

In [GMS, p.89–90], Serre proves that $\text{Rat}(G/\mathbb{Q})$ is false for $G = \widehat{S}_4$ and \widehat{S}_5 ; actually he proves a bit more. From Serre’s proof it is easy to find that $\mathbb{Q}(\widehat{S}_4)$ and $\mathbb{Q}(\widehat{S}_5)$ are not retract \mathbb{Q} -rational (see [Ka2, Section 1] for the relationship of the property $\text{Rat}(G/k)$ and the retract k -rationality of $k(G)$). This is the reason why we formulate Serre’s Theorem in the above version. In fact, Theorem 1.2 can be perceived also from Serre’s own remark in [GMS, p.13, Remark 5.8].

On the other hand, Plans proved the following result.

Theorem 1.3 (Plans [Pl1; Pl2]) (1) For any field k , $k(\widetilde{S}_4)$ is k -rational. Thus, if k is a field with $\text{char } k = 0$, $k(\widetilde{S}_5)$ is also k -rational.

(2) For any field k with $\text{char } k = 0$ such that $\sqrt{-1} \in k$, both $k(\widehat{S}_4)$ and $k(\widehat{S}_5)$ are k -rational.

The main result of this article is the following rationality criterion for $k(\widehat{S}_4)$ and $k(\widehat{S}_5)$.

Theorem 1.4 *Let k be a field with $\text{char } k \neq 2$ or 3 , and ζ_8 be a primitive 8-th root of unity in some extension field of k . If $k(\zeta_8)$ is a cyclic extension of k , then $k(\widehat{S}_4)$ is k -rational; if it is assumed furthermore that $\text{char } k = 0$, then $k(\widehat{S}_5)$ is also k -rational.*

When k is a field with $\text{char } k = p > 0$ and $p \neq 2$, the assumption that $k(\zeta_8)$ is a cyclic extension of k is satisfied automatically.

We don't know whether Theorem 1.2 is valid for fields k other than the field \mathbb{Q} , for examples, some field k satisfying the condition that $k(\zeta_8)$ is not cyclic over k . On the other hand, Serre shows that $\mathbb{Q}(G)$ is not retract \mathbb{Q} -rational if G is any one of the groups \widehat{S}_4 , \widehat{S}_5 , $SL_2(\mathbb{F}_7)$, $SL_2(\mathbb{F}_9)$ and the generalized quaternion group of order 16 (see [GMS, p.90, Example 33.27]). Besides the cases \widehat{S}_4 and \widehat{S}_5 studied in Theorem 1.4, it is known that $k(G)$ is k -rational provided that G is the generalized quaternion group of order 16 and $k(\zeta_8)$ is cyclic over k [Ka1]. We don't know whether analogous results as Theorem 1.4 are valid when the groups are $SL_2(\mathbb{F}_7)$ and $SL_2(\mathbb{F}_9)$.

The main idea of the proof of Theorem 1.4 is by applying the method of Galois descent, namely we enlarge the field k to $k(\zeta_8)$ first, solve the rationality of $k(\zeta_8)(\widehat{S}_4)$, and then descend the ground field of $k(\zeta_8)(\widehat{S}_4)$ to k . This will finish the proof of the rationality of $k(\widehat{S}_4)$. By Plans's Theorem (see Theorem 2.5), $k(\widehat{S}_5)$ is a rational extension of $k(\widehat{S}_4)$. Hence $k(\widehat{S}_5)$ is k -rational also.

In showing that $k(\zeta_8)(\widehat{S}_4)$ is $k(\zeta_8)$ -rational and $k(\widehat{S}_4)$ is k -rational, we will construct a 4-dimensional faithful representation V of \widehat{S}_4 defined over the field k . Although it is not very difficult to find such a 4-dimensional representation, it seems the representation and the idea to find it are not well-known. Once we have this representation, write $\pi = \text{Gal}(k(\zeta_8)/k)$. By Theorem 2.2 of this paper, it is easy to see that $k(\widehat{S}_4)$ is rational over $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$. Thus it remains to prove $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is k -rational.

The rationality problem of $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is not an easy job. It requires special efforts and lots of computations. In several steps we use computers to facilitate the process of symbolic computation, because computers can save us from the laborious manual computation. We emphasize computers play only a minor role in the above sense; we don't use particular codes of data bases, e.g. GAP etc. We will point out that the first several steps in proving $k(\zeta_8)(V)^{\langle \widehat{S}_4, \pi \rangle}$ is k -rational are rather similar to those in [KZ, Section 5]. This is not surprising because we deal with \widetilde{S}_4 in [KZ, Section 5] and the groups \widehat{S}_4 and \widetilde{S}_4 have a common subgroup \widetilde{A}_4 .

We organize this paper as follows. We recall some preliminaries in Section 2, which will be used in the proof of Theorem 1.4. In Section 3, several low-dimensional faithful representations of \widehat{S}_4 over a field k with $\text{char } k \neq 2$ will be constructed (the reader may find another explicit construction in [Kar, p.177-179]). Theorem 1.4 will be proved in Section 4. In Section 5 we will consider the rationality problem of $k(G_n)$ (see Definition 5.1 for the group G_n).

Throughout this article, whenever we write $k(x_1, x_2, x_3, x_4)$ or $k(x, y)$ without explanation, it is understood that it is a rational function field over k . We will denote ζ_8 (or simply ζ) a primitive 8-th root of unity.

§2. Preliminaries

We recall several results which will be used in tackling the rationality problem.

Theorem 2.1 (Ahmad, Hajja and Kang [AHK, Theorem 3.1]) *Let L be any field, $L(x)$ the rational function field of one variable over L and G a finite group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$ where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G, \deg g(x) \geq 1\}$, any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property $L(x)^G = L^G(f)$.*

Theorem 2.2 (Hajja and Kang [HK, Theorem 1]) *Let G be a finite group acting on $L(x_1, \dots, x_n)$, the rational function field of n variables over a field L . Suppose that*

- (i) *for any $\sigma \in G$, $\sigma(L) \subset L$;*
- (ii) *the restriction of the action of G to L is faithful;*
- (iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is a $n \times 1$ matrix over L .

Then there exist elements $z_1, \dots, z_n \in L(x_1, \dots, x_n)$ which are algebraically independent over L , and $L(x_1, \dots, x_n) = L(z_1, \dots, z_n)$ so that $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq n$.

Theorem 2.3 (Yamasaki [Ya]) *Let k be a field with $\text{char } k \neq 2$, $a \in k \setminus \{0\}$, σ be a k -automorphism of the rational function field $k(x, y)$ defined by $\sigma(x) = a/x$, $\sigma(y) = a/y$. Then $k(x, y)^{\langle \sigma \rangle} = k(u, v)$ where $u = (x - y)/(a - xy)$, $v = (x + y)/(a + xy)$.*

Theorem 2.4 (Masuda [Ma, Theorem 3; HoK, Theorem 2.2]) *Let k be any field, σ be a k -automorphism of the rational function field $k(x, y, z)$ defined by $\sigma : x \mapsto y \mapsto z \mapsto x$. Then $k(x, y, z)^{\langle \sigma \rangle} = k(s_1, u, v) = k(s_3, u, v)$ where s_1, s_2, s_3 are the elementary symmetric functions of degree one, two, three in x, y, z and u and v are defined as*

$$u = \frac{x^2y + y^2z + z^2x - 3xyz}{x^2 + y^2 + z^2 - xy - yz - zx},$$

$$v = \frac{xy^2 + yz^2 + zx^2 - 3xyz}{x^2 + y^2 + z^2 - xy - yz - zx}.$$

Theorem 2.5 (Plans [Pl2, Theorem 11]) *Let $n \geq 5$ be an odd integer and k be a field with $\text{char } k = 0$. Then $k(\widehat{S_n})$ is rational over $k(\widehat{S_{n-1}})$.*

Theorem 2.6 (Kang and Plans [KP, Theorem 1.9]) *Let k be any field, G_1 and G_2 be two finite groups. If both $k(G_1)$ and $k(G_2)$ are k -rational, so is $k(G_1 \times G_2)$.*

§3. Faithful representations of \widehat{S}_4

In this section and the next section, the field k we consider is of char $k \neq 2$ or 3 . We will denote by $\zeta_8 = (1 + \sqrt{-1})/\sqrt{2}$, a primitive 8-th root of unity.

In [Sp, p.92] a generating set of \widehat{S}_4 is given (where the group is called the binary octahedral group) : $\widehat{S}_4 = \langle a', b, c \rangle$ with relations $a'^8 = b^4 = c^6 = 1$, $ba'b^{-1} = a'^{-1}$, $cbc^{-1} = a'^2$, $(a'c)^2 = -a'^2b$ (here -1 is the element which is equal to $a'^4 = b^2 = c^3$). Note that we have a short exact sequence of groups

$$1 \rightarrow \{\pm 1\} \rightarrow \widehat{S}_4 \xrightarrow{p} S_4 \rightarrow 1$$

and $p(a') = (1, 2, 3, 4)$, $p(b) = (1, 4)(2, 3)$, $p(c) = (1, 2, 3)$. Note that $p(ba') = (1, 4)(2, 3)(1, 2, 3, 4) = (1, 3)$.

If $\zeta_8 \in k$, a faithful 2-dimensional representation $\Phi : \widehat{S}_4 \rightarrow GL_2(k)$ is given in [Sp, p.92] as follows (we write $\zeta = \zeta_8$),

$$(3.1) \quad \Phi(a') = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^7 \end{pmatrix}, \quad \Phi(b) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \Phi(c) = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^7 & \zeta^7 \\ \zeta^5 & \zeta \end{pmatrix}.$$

Suppose that $\sqrt{2} \in k$ (but it is unnecessary that $\sqrt{-1} \in k$). We may obtain a 4-dimensional representation $\widehat{S}_4 \rightarrow GL_4(k)$ by substituting

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

for $\sqrt{-1}$, α (where k_0 is the prime field of k and $\alpha \in k_0(\sqrt{2})$) in Formula (3.1). This process is just an easy application of Weil's restriction [We; Vo, p.38]. Thus we get

$$(3.2) \quad \begin{aligned} a' &\mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & & \\ 1 & 1 & & \\ & & 1 & 1 \\ & & -1 & 1 \end{pmatrix}, & b &\mapsto \begin{pmatrix} & & -1 & \\ & 1 & & \\ -1 & & & \\ 1 & & & \end{pmatrix}, \\ c &\mapsto \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Similarly, when $\sqrt{-2} \in k$ (but it may happen that $\sqrt{-1} \notin k$), write $\sqrt{-2} = \sqrt{-1} \cdot \sqrt{2}$. Thus represent $\sqrt{2}$ as $-\sqrt{-1} \cdot \sqrt{-2}$ and $\zeta = (1 + \sqrt{-1})/\sqrt{2}$ becomes $\sqrt{-2}(1 - \sqrt{-1})/2$. Substitute

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

for $\sqrt{-1}$, α (where k_0 is the prime field of k and $\alpha \in k_0(\sqrt{-2})$) in Formula (3.1). We get

$$(3.3) \quad \begin{aligned} a' &\mapsto \frac{\sqrt{-2}}{2} \begin{pmatrix} 1 & 1 & \vdots \\ -1 & 1 & \vdots \\ \hdashline & & \\ & -1 & 1 \\ & -1 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} & & -1 \\ & 1 & \\ & -1 & \\ 1 & & \end{pmatrix}, \\ c &\mapsto \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

By the same way, if $\sqrt{-1} \in k$ (but it may happen that $\sqrt{2} \notin k$), substitute

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

for $\sqrt{2}$, α (where k_0 is the prime field of k and $\alpha \in k_0(\sqrt{-1})$) in Formula (3.1). We get

$$(3.4) \quad \begin{aligned} a' &\mapsto \begin{pmatrix} 0 & 1 + \sqrt{-1} & \vdots \\ (1 + \sqrt{-1})/2 & 0 & \vdots \\ \hdashline & & \\ & 0 & 1 - \sqrt{-1} \\ & (1 - \sqrt{-1})/2 & 0 \end{pmatrix}, \\ b &\mapsto \begin{pmatrix} & \sqrt{-1} & 0 \\ & 0 & \sqrt{-1} \\ \hdashline & & \\ \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \\ c &\mapsto \begin{pmatrix} (1 - \sqrt{-1})/2 & 0 & (1 - \sqrt{-1})/2 & 0 \\ 0 & (1 - \sqrt{-1})/2 & 0 & (1 - \sqrt{-1})/2 \\ (-1 - \sqrt{-1})/2 & 0 & (1 + \sqrt{-1})/2 & 0 \\ 0 & (-1 - \sqrt{-1})/2 & 0 & (1 + \sqrt{-1})/2 \end{pmatrix}. \end{aligned}$$

Finally, from Formula (3.2) we may get a faithful 8-dimensional representation of \widehat{S}_4 into $GL_8(k_0)$ where k_0 is the prime field of k . Explicitly, substitute

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

for $\sqrt{2}$, α where $\alpha \in k_0$ in Formula (3.2). We get

$$\begin{aligned}
 (3.5) \quad & a' \mapsto \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 & -2 & \vdots \\ 1 & 0 & -1 & 0 & \\ 0 & 2 & 0 & 2 & \\ 1 & 0 & 1 & 0 & \\ \hline & & & & 0 & 2 & 0 & 2 \\ & & & & 1 & 0 & 1 & 0 \\ & & & & 0 & -2 & 0 & 2 \\ & & & & -1 & 0 & 1 & 0 \end{pmatrix}, \\
 & b \mapsto \begin{pmatrix} & & & & \vdots & & -1 & 0 \\ & & & & & & 0 & -1 \\ & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \\ \hline & & -1 & 0 & & & & \\ & & 0 & -1 & & & & \\ 1 & 0 & & & \vdots & & & \\ 0 & 1 & & & & & & \end{pmatrix}, \\
 & c \mapsto \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 & \vdots & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & & 0 & -1 & 0 & 1 \\ \hline -1 & 0 & 1 & 0 & & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & & 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & & 0 & 1 & 0 & 1 \end{pmatrix}.
 \end{aligned}$$

§4. Proof of Theorem 1.4

By Theorem 2.5, in case $\text{char } k = 0$ and it is known that $k(\widehat{S}_4)$ is k -rational, it follows immediate that $k(\widehat{S}_5)$ is also k -rational. Hence, in proving Theorem 1.4, it suffices to prove the rationality of $k(\widehat{S}_4)$.

By assumption, $k(\zeta_8)$ is a cyclic extension of k . Hence at least one of $\sqrt{-1}$, $\sqrt{2}$, $\sqrt{-2}$ belongs to k .

Case 1. $\zeta_8 \in k$.

Since $\text{char } k \neq 2$ or 3 , the group algebra $k[\widehat{S}_4]$ is semi-simple. Hence the 2-dimensional faithful representation provided by Formula (3.1) can be embedded into the regular representation whose dual space is $V_{\text{reg}} = \bigoplus_{g \in \widehat{S}_4} k \cdot x(g)$ where \widehat{S}_4 acts on V_{reg} by $h \cdot x(g) = x(hg)$ for any $g, h \in \widehat{S}_4$. Applying Theorem 2.2, we find $k(\widehat{S}_4) = k(x(g) :$

$g \in \widehat{S}_4^{\widehat{S}_4}$ is rational over $k(x, y)^{\widehat{S}_4}$ where the actions given by Formula (3.1) are as follows

$$\begin{aligned} a' : x &\mapsto \zeta x, \quad y \mapsto \zeta^7 y, \\ b : x &\mapsto \sqrt{-1}y, \quad y \mapsto \sqrt{-1}x, \\ c : x &\mapsto (\zeta^7 x + \zeta^5 y)/\sqrt{2}, \quad y \mapsto (\zeta^7 x + \zeta y)/\sqrt{2}. \end{aligned}$$

Define $z = x/y$. Then $k(x, y) = k(z, x)$. Apply Theorem 2.1. We get $k(z, x)^{\widehat{S}_4} = k(z)^{\widehat{S}_4}(t)$ for some element t fixed by \widehat{S}_4 . The field $k(z)^{\widehat{S}_4}$ is k -rational by Lüroth's Theorem. Hence $k(z, x)^{\widehat{S}_4}$ and $k(\widehat{S}_4)$ are k -rational.

Case 2. $\sqrt{2} \in k$, but $\sqrt{-1} \notin k$.

We will use the 4-dimensional faithful representation of \widehat{S}_4 over k provided by Formula (3.2). This representation provides an action of \widehat{S}_4 on $k(x_1, x_2, x_3, x_4)$ given by

$$\begin{aligned} (4.1) \quad a' : x_1 &\mapsto (x_1 + x_2)/\sqrt{2}, \quad x_2 \mapsto (-x_1 + x_2)/\sqrt{2}, \quad x_3 \mapsto (x_3 - x_4)/\sqrt{2}, \\ &\quad x_4 \mapsto (x_3 + x_4)/\sqrt{2}, \\ b : x_1 &\mapsto x_4 \mapsto -x_1, \quad x_2 \mapsto -x_3, \quad x_3 \mapsto x_2, \\ c : x_1 &\mapsto (x_1 - x_2 - x_3 - x_4)/2, \quad x_2 \mapsto (x_1 + x_2 + x_3 - x_4)/2, \\ &\quad x_3 \mapsto (x_1 - x_2 + x_3 + x_4)/2, \quad x_4 \mapsto (x_1 + x_2 - x_3 + x_4)/2. \end{aligned}$$

Step 1. Apply Theorem 2.2 and use the same arguments in Case 1. We find that $k(\widehat{S}_4)$ is rational over $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$. It remains to show that $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$ is k -rational.

Step 2. Write $\pi = \text{Gal}(k(\sqrt{-1})/k) = \langle \rho \rangle$ where $\rho(\sqrt{-1}) = -\sqrt{-1}$.

We extend the actions of π and \widehat{S}_4 on $k(\sqrt{-1})$ and $k(x_1, x_2, x_3, x_4)$ to $k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by requiring that $\rho(x_i) = x_i$ for $1 \leq i \leq 4$ and $g(\sqrt{-1}) = \sqrt{-1}$ for all $g \in \widehat{S}_4$. It follows that $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4} = \{k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{\langle \rho \rangle}\}^{\langle a', b, c \rangle} = k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{\langle a', b, c, \rho \rangle}$.

Define $y_1, y_2, y_3, y_4 \in k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by

$$\begin{aligned} y_1 &= \sqrt{-1}x_1 + \sqrt{-1}x_2 - x_3 + x_4, & y_2 &= -\sqrt{-1}x_1 + \sqrt{-1}x_2 + x_3 + x_4, \\ y_3 &= x_1 - x_2 - \sqrt{-1}x_3 - \sqrt{-1}x_4, & y_4 &= x_1 + x_2 - \sqrt{-1}x_3 + \sqrt{-1}x_4. \end{aligned}$$

Then $k(\sqrt{-1})(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(y_1, y_2, y_3, y_4)$ and the actions in Formula

(4.1) becomes

$$\begin{aligned}
(4.2) \quad & a' : y_1 \mapsto (y_1 + y_2)/\sqrt{2}, \quad y_2 \mapsto (-y_1 + y_2)/\sqrt{2}, \quad y_3 \mapsto (y_3 + y_4)/\sqrt{2}, \\
& \quad y_4 \mapsto (-y_3 + y_4)/\sqrt{2}, \\
& b : y_1 \mapsto \sqrt{-1}y_1, \quad y_2 \mapsto -\sqrt{-1}y_2, \quad y_3 \mapsto \sqrt{-1}y_3, \quad y_4 \mapsto -\sqrt{-1}y_4, \\
& c : y_1 \mapsto (y_1 - \sqrt{-1}y_2)/(1 + \sqrt{-1}), \quad y_2 \mapsto (y_1 + \sqrt{-1}y_2)/(1 + \sqrt{-1}), \\
& \quad y_3 \mapsto (y_3 - \sqrt{-1}y_4)/(1 + \sqrt{-1}), \quad y_4 \mapsto (y_3 + \sqrt{-1}y_4)/(1 + \sqrt{-1}), \\
& \rho : y_1 \mapsto -\sqrt{-1}y_4, \quad y_2 \mapsto \sqrt{-1}y_3, \quad y_3 \mapsto \sqrt{-1}y_2, \quad y_4 \mapsto -\sqrt{-1}y_1.
\end{aligned}$$

Note that the action of a'^2 is given by

$$a'^2 : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3.$$

It may be interesting if the reader is willing to compare the actions in Formula (4.2) with those in [KZ, Section 4]. It turns out that the formulae for b , a'^2 , c^2 are completely the same as those for λ_1 , λ_2 , σ in [KZ, Formula (4.3)]. As mentioned before, both the subgroups $\langle b, a'^2, c^2 \rangle$ and $\langle \lambda_1, \lambda_2, \sigma \rangle$ are isomorphic to \widetilde{A}_4 ($\widetilde{A}_4 = p^{-1}(A_4)$) in the notation of Section 3) as abstract groups.

Step 3. Before we find $k(\sqrt{-1})(y_1, y_2, y_3, y_4)^{\langle \widehat{S}_4, \pi \rangle}$, we will find $k(\sqrt{-1})(y_1, y_2, y_3, y_4)^{\langle b, a'^2 \rangle}$ first. The method is the same as Step 3 and Step 4 in [KZ, Section 4]. We will write down the details, for the convenience of the reader.

Define $z_1 = y_1/y_2$, $z_2 = y_3/y_4$, $z_3 = y_1/y_3$. By Theorem 2.1, we find that $k(\sqrt{-1})(y_1, y_2, y_3, y_4)^{\langle \widehat{S}_4, \pi \rangle} = k(\sqrt{-1})(z_1, z_2, z_3)(y_4)^{\langle \widehat{S}_4, \pi \rangle} = k(\sqrt{-1})(z_1, z_2, z_3)^{\langle \widehat{S}_4, \pi \rangle}(z_0)$ where z_0 is fixed by the actions of \widehat{S}_4 and π . It remains to show that $k(\sqrt{-1})(z_1, z_2, z_3)^{\langle \widehat{S}_4, \pi \rangle}$ is k -rational.

Define $u_1 = z_1/z_2$, $u_2 = z_1z_2$, $u_3 = z_3$. Then $k(\sqrt{-1})(z_1, z_2, z_3)^{\langle b \rangle} = k(\sqrt{-1})(u_1, u_2, u_3)$. The action of a'^2 is given by

$$a'^2 : u_1 \mapsto 1/u_1, \quad u_2 \mapsto 1/u_2, \quad u_3 \mapsto u_3/u_1.$$

Define $v_1 = (u_1 - u_2)/(1 - u_1u_2)$, $v_2 = (u_1 + u_2)/(1 + u_1u_2)$, $v_3 = u_3(1 + (1/u_1))$. Then $k(\sqrt{-1})(u_1, u_2, u_3)^{\langle a'^2 \rangle} = k(\sqrt{-1})(u_1, u_2, v_3)^{\langle a'^2 \rangle} = k(\sqrt{-1})(v_1, v_2, v_3)$ by Theorem 2.3 (note that $a'^2(v_3) = v_3$). In summary, $k(\sqrt{-1})(z_1, z_2, z_3)^{\langle b, a'^2 \rangle} = k(\sqrt{-1})(v_1, v_2, v_3)$.

Step 4. The action of c on v_1, v_2, v_3 is given by

$$c : v_1 \mapsto 1/v_2, \quad v_2 \mapsto v_1/v_2, \quad v_3 \mapsto v_3(v_1 + v_2)/[v_2(1 + v_1)].$$

Define $X_3 = v_3(1 + v_1 + v_2)/[(1 + v_1)(1 + v_2)]$. Then $c(X_3) = X_3$ and $k(\sqrt{-1})(v_1, v_2, v_3) = k(\sqrt{-1})(v_1, v_2, X_3)$. Thus we may apply Theorem 2.4 (regarding $v_1, 1/v_2, v_2/v_1$ as x, y, z in Theorem 2.4). More precisely, define

$$\begin{aligned}
X_1 &= (v_1^3v_2^3 + v_1^3 + v_2^3 - 3v_1^2v_2^2)/(v_1^4v_2^2 + v_2^4 + v_1^2 - v_1^2v_2^3 - v_1v_2^2 - v_1^3v_2), \\
X_2 &= (v_1v_2^4 + v_1v_2 + v_1^4v_2 - 3v_1^2v_2^2)/(v_1^4v_2^2 + v_2^4 + v_1^2 - v_1^2v_2^3 - v_1v_2^2 - v_1^3v_2).
\end{aligned}$$

By Theorem 2.4 we get $k(\sqrt{-1})(v_1, v_2, X_3)^{(c)} = k(\sqrt{-1})(X_1, X_2, X_3)$.

Step 5. With the aid of computers, the actions of a' and ρ on X_1, X_2, X_3 are given by

$$\begin{aligned} a' : X_1 &\mapsto X_1/(X_1^2 - X_1X_2 + X_2^2), \quad X_2 \mapsto X_2/(X_1^2 - X_1X_2 + X_2^2), \quad X_3 \mapsto X_3, \\ \rho : X_1 &\mapsto X_2/(X_1^2 - X_1X_2 + X_2^2), \quad X_2 \mapsto X_1/(X_1^2 - X_1X_2 + X_2^2), \quad X_3 \mapsto -2A/X_3 \end{aligned}$$

where $A = g_1g_2g_3^{-1}$ and

$$\begin{aligned} g_1 &= (1 + X_1)^2 - X_2(1 + X_1) + X_2^2, \quad g_2 = (1 + X_2)^2 - X_1(1 + X_2) + X_1^2, \\ g_3 &= 1 + X_1 + X_2 + X_1^3 + X_2^3 + X_1X_2(3X_1X_2 - 2X_1^2 - 2X_2^2 + 2) + X_1^4 + X_2^4. \end{aligned}$$

Note that $\rho(g_1) = g_2/(X_1^2 - X_1X_2 + X_2^2)$.

Define $Y_1 = X_1/X_2, Y_2 = X_1, Y_3 = X_1X_3/g_1$. We find that

$$a' : Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_1^2/(Y_2(1 - Y_1 + Y_1^2)), \quad Y_3 \mapsto Y_3.$$

Thus $k(\sqrt{-1})(X_1, X_2, X_3)^{(a')} = k(\sqrt{-1})(Y_1, Y_2, Y_3)^{(a')} = k(\sqrt{-1})(Z_1, Z_2, Z_3)$ where $Z_1 = Y_1, Z_2 = Y_2 + a'(Y_2), Z_3 = Y_3$.

Step 6. Using computers, we find that the action of ρ is given by

$$\rho : Z_1 \mapsto 1/Z_1, \quad Z_2 \mapsto Z_2/Z_1, \quad Z_3 \mapsto -2Z_1^3/(A'Z_3)$$

where $A' = -2Z_1^2 + Z_1Z_2 + Z_2^2 + 4Z_1^3 - 2Z_1Z_2^2 - 2Z_1^4 + 3Z_1^2Z_2^2 + Z_1^4Z_2 - 2Z_1^3Z_2^2 + Z_1^4Z_2^2$.

Define $U_1 = Z_2 + \rho(Z_2), U_2 = \sqrt{-1}(Z_2 - \rho(Z_2)), U_3 = Z_3 + \rho(Z_3), U_4 = \sqrt{-1}(Z_3 - \rho(Z_3))$. It is easy to verify that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)} = k(U_1, U_2, U_3, U_4)$ with a relation

$$U_3^2 + U_4^2 + 32(U_1^2 + U_2^2)/B = 0$$

where $B = (U_1^2 - 3U_2^2)^2 + 4U_1(U_1^2 - 3U_2^2) + 32U_2^2$.

Divide the above relation by $16(U_1^2 + U_2^2)^2/B^2$. We get

$$(BU_3/(4U_1^2 + 4U_2^2))^2 + (BU_4/(4U_1^2 + 4U_2^2))^2 + 2B/(U_1^2 + U_2^2) = 0.$$

Multiply this relation by $U_1^2 + U_2^2$ and use the identity $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) = (\alpha\delta + \beta\gamma)^2 + (\alpha\gamma - \beta\delta)^2$. The relation is simplified as

$$(4.3) \quad V_3^2 + V_4^2 + 2B = 0$$

where $V_3 = B(U_1U_3 + U_2U_4)/(4U_1^2 + 4U_2^2), V_4 = B(U_1U_4 - U_2U_3)/(4U_1^2 + 4U_2^2)$. Note that $k(U_1, U_2, U_3, U_4) = K(U_1, U_2, V_3, V_4)$.

Define $w_1 = 8U_1/(U_1^2 - 3U_2^2), w_2 = 8U_2/(U_1^2 - 3U_2^2), w_3 = V_3/(U_1^2 - 3U_2^2), w_4 = V_4/(U_1^2 - 3U_2^2)$. Then $k(U_1, U_2, V_3, V_4) = k(w_1, w_2, w_3, w_4)$ and the relation (4.3) becomes

$$w_3^2 + w_4^2 + 2 + w_1 + w_2^2 = 0.$$

Hence $w_1 \in k(w_2, w_3, w_4)$. Thus $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{\langle \rho \rangle} = k(w_2, w_3, w_4)$ is k -rational. Done.

Case 3. $\sqrt{-2} \in k$, but $\sqrt{-1} \notin k$.

We use the 4-dimensional faithful representation of \widehat{S}_4 over k provided by Formula (3.3). This representation provides an action of \widehat{S}_4 on $k(x_1, x_2, x_3, x_4)$ given by

$$\begin{aligned} a' : x_1 &\mapsto \sqrt{-2}(x_1 - x_2)/2, \quad x_2 \mapsto \sqrt{-2}(x_1 + x_2)/2, \quad x_3 \mapsto \sqrt{-2}(-x_3 - x_4)/2, \\ &\quad x_4 \mapsto \sqrt{-2}(x_3 - x_4)/2, \\ b : x_1 &\mapsto x_4 \mapsto -x_1, \quad x_2 \mapsto -x_3, \quad x_3 \mapsto x_2, \\ c : x_1 &\mapsto (x_1 - x_2 - x_3 - x_4)/2, \quad x_2 \mapsto (x_1 + x_2 + x_3 - x_4)/2, \\ &\quad x_3 \mapsto (x_1 - x_2 + x_3 + x_4)/2, \quad x_4 \mapsto (x_1 + x_2 - x_3 + x_4)/2. \end{aligned}$$

The proof of this case is very similar to that of Case 2.

Step 1. Apply Theorem 2.2. We find that $k(\widehat{S}_4)$ is rational over $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$. Hence the proof is reduced to proving $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4}$ is k -rational.

Step 2. Write $\pi = \text{Gal}(k(\sqrt{-1})/k) = \langle \rho \rangle$ where $\rho(\sqrt{-1}) = -\sqrt{-1}$.

Extend the actions of π and \widehat{S}_4 to $k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ as in Step 2 of Case 2. We find that $k(x_1, x_2, x_3, x_4)^{\widehat{S}_4} = k(\sqrt{-1})(x_1, x_2, x_3, x_4)^{\langle a', b, c, \rho \rangle}$.

Define $y_1, y_2, y_3, y_4 \in k(\sqrt{-1})(x_1, x_2, x_3, x_4)$ by

$$\begin{aligned} y_1 &= -x_1 - \sqrt{-1}x_2 + x_3 + \sqrt{-1}x_4, & y_2 &= \sqrt{-1}x_1 - x_2 + \sqrt{-1}x_3 - x_4, \\ y_3 &= x_1 - \sqrt{-1}x_2 + x_3 - \sqrt{-1}x_4, & y_4 &= \sqrt{-1}x_1 + x_2 - \sqrt{-1}x_3 - x_4. \end{aligned}$$

We get $k(\sqrt{-1})(x_1, x_2, x_3, x_4) = k(\sqrt{-1})(y_1, y_2, y_3, y_4)$ and the actions are

$$\begin{aligned} (4.4) \quad a' : y_1 &\mapsto (-y_1 - y_2)/\sqrt{2}, \quad y_2 \mapsto (y_1 - y_2)/\sqrt{2}, \quad y_3 \mapsto (y_3 + y_4)/\sqrt{2}, \\ &\quad y_4 \mapsto (-y_3 + y_4)/\sqrt{2}, \\ b : y_1 &\mapsto \sqrt{-1}y_1, \quad y_2 \mapsto -\sqrt{-1}y_2, \quad y_3 \mapsto \sqrt{-1}y_3, \quad y_4 \mapsto -\sqrt{-1}y_4, \\ c : y_1 &\mapsto (y_1 - \sqrt{-1}y_2)/(1 + \sqrt{-1}), \quad y_2 \mapsto (y_1 + \sqrt{-1}y_2)/(1 + \sqrt{-1}), \\ &\quad y_3 \mapsto (y_3 - \sqrt{-1}y_4)/(1 + \sqrt{-1}), \quad y_4 \mapsto (y_3 + \sqrt{-1}y_4)/(1 + \sqrt{-1}), \\ \rho : y_1 &\mapsto \sqrt{-1}y_4, \quad y_2 \mapsto -\sqrt{-1}y_3, \quad y_3 \mapsto -\sqrt{-1}y_2, \quad y_4 \mapsto \sqrt{-1}y_1. \end{aligned}$$

Note that the action of a'^2 is

$$a'^2 : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3.$$

Compare Formula (4.2) and Formula (4.4). The actions of a'^2 , b , c in both cases are the same.

Step 3. Define $z_1 = y_1/y_2$, $z_2 = y_3/y_4$, $z_3 = y_1/y_3$. As in Step 3 of Case 2, it suffices to prove $k(\sqrt{-1})(z_1, z_2, z_3)^{\langle \widehat{S_4}, \pi \rangle}$ is k -rational.

Define $u_1, u_2, u_3, v_1, v_2, v_3, X_1, X_2, X_3$ by the same formulae as in Step 3 and Step 4 of Case 2. We find that $k(\sqrt{-1})(z_1, z_2, z_3)^{\langle b, a'^2, c \rangle} = k(\sqrt{-1})(X_1, X_2, X_3)$.

Step 4. The actions of a' , ρ on X_1, X_2, X_3 are slightly different from Step 5 of Case 2. In the present case, we have

$$\begin{aligned} a' : X_1 &\mapsto X_1/(X_1^2 - X_1X_2 + X_2^2), \quad X_2 \mapsto X_2/(X_1^2 - X_1X_2 + X_2^2), \quad X_3 \mapsto -X_3, \\ \rho : X_1 &\mapsto X_2/(X_1^2 - X_1X_2 + X_2^2), \quad X_2 \mapsto X_1/(X_1^2 - X_1X_2 + X_2^2), \quad X_3 \mapsto -2A/X_3 \end{aligned}$$

where $A = g_1g_2g_3^{-1}$ and

$$\begin{aligned} g_1 &= (1 + X_1)^2 - X_2(1 + X_1) + X_2^2, \quad g_2 = (1 + X_2)^2 - X_1(1 + X_2) + X_1^2, \\ g_3 &= 1 + X_1 + X_2 + X_1^3 + X_2^3 + X_1X_2(3X_1X_2 - 2X_1^2 - 2X_2^2 + 2) + X_1^4 + X_2^4. \end{aligned}$$

Note that the action of ρ is the same as in Step 5 of Case 2.

Define $Y_1 = X_1/X_2$, $Y_2 = X_1$, $Y_3 = X_1X_3/g_1$. We get

$$a' : Y_1 \mapsto Y_1, \quad Y_2 \mapsto Y_1^2/(Y_2(1 - Y_1 + Y_1^2)), \quad Y_3 \mapsto -Y_3.$$

Thus $k(\sqrt{-1})(X_1, X_2, X_3)^{\langle a' \rangle} = k(\sqrt{-1})(Y_1, Y_2, Y_3)^{\langle a' \rangle} = k(\sqrt{-1})(Z_1, Z_2, Z_3)$ where $Z_1 = Y_1$, $Z_2 = Y_2 + a'(Y_2)$, $Z_3 = Y_3(Y_2 - a'(Y_2))$.

Step 5. Using computers, we find that the action of ρ is given by

$$\rho : Z_1 \mapsto 1/Z_1, \quad Z_2 \mapsto Z_2/Z_1, \quad Z_3 \mapsto C/Z_3$$

where $C = 2Z_1^2(-4Z_1^2 + Z_2^2 - Z_1Z_2^2 + Z_1^2Z_2^2)/[(1 - Z_1 + Z_1^2)(-2Z_1^2 + Z_1Z_2 + Z_2^2 + 4Z_1^3 - 2Z_1Z_2^2 - 2Z_1^4 + 3Z_1^2Z_2^2 + Z_1^4Z_2 - 2Z_1^3Z_2^2 + Z_1^4Z_2^2)]$.

Define $U_1 = Z_2 + \rho(Z_2)$, $U_2 = \sqrt{-1}(Z_2 - \rho(Z_2))$, $U_3 = Z_3 + \rho(Z_3)$, $U_4 = \sqrt{-1}(Z_3 - \rho(Z_3))$. We find that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{\langle \rho \rangle} = k(U_1, U_2, U_3, U_4)$ with a relation

$$(4.5) \quad U_3^2 + U_4^2 = 8(U_1^2 + U_2^2)^2(-16 + U_1^2 - 3U_2^2)/(B(U_1^2 - 3U_2^2))$$

where $B = (U_1^2 - 3U_2^2)^2 + 4U_1(U_1^2 - 3U_2^2) + 32U_2^2$.

Note that the above formula of B is identically the same as that in Step 6 of Case 2.

It remains to simplify the relation in Formula (4.5).

Divide both sides of Formula (4.5) by $(U_1^2 + U_2^2)^2$. We get

$$(U_3/(U_1^2 + U_2^2))^2 + (U_4/(U_1^2 + U_2^2))^2 = 8(-16 + U_1^2 - 3U_2^2)/(B(U_1^2 - 3U_2^2)).$$

Divide both sides of the above identity by $(2(U_1^2 - 3U_2^2)/B)^2$. We get a relation

$$(4.6) \quad V_3^2 + V_4^2 = 2(1 - V_1^2 + 3V_2^2)(1 + V_1 + 2V_2^2)$$

where $V_1 = 4U_1/(U_1^2 - 3U_2^2)$, $V_2 = 4U_2/(U_1^2 - 3U_2^2)$, $V_3 = BU_3/((U_1^2 - 3U_2^2)(2U_1^2 + 2U_2^2))$, $V_4 = BU_4/((U_1^2 - 3U_2^2)(2U_1^2 + 2U_2^2))$.

Note that $k(U_1, U_2, U_3, U_4) = k(V_1, V_2, V_3, V_4)$.

Define $w_1 = 1/(1 + V_1)$, $w_2 = V_2/(1 + V_1)$, $w_3 = V_3/(1 + V_1)^2$, $w_4 = V_4/(1 + V_1)^2$. We get $k(V_1, V_2, V_3, V_4) = k(w_1, w_2, w_3, w_4)$ and the relation (4.6) becomes

$$w_3^2 + w_4^2 = 2(-1 + 2w_1 + 3w_2^2)(w_1 + 2w_2^2).$$

Divide the above identity by $(w_1 + 2w_2^2)^2$. We get

$$(w_3/(w_1 + 2w_2^2))^2 + (w_4/(w_1 + 2w_2^2))^2 = 2(-1 + 2w_1 + 3w_2^2)/(w_1 + 2w_2^2).$$

Since $2(-1 + 2w_1 + 3w_2^2)/(w_1 + 2w_2^2)$ is a “fractional linear transformation” of w_1 and it belongs to $k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2))$, we find $w_1 \in k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2))$. Thus $k(w_1, w_2, w_3, w_4) = k(w_2, w_3/(w_1 + 2w_2^2), w_4/(w_1 + 2w_2^2))$. We find that $k(\sqrt{-1})(Z_1, Z_2, Z_3)^{(\rho)}$ is k -rational.

Case 4. $\sqrt{-1} \in k$, but $\sqrt{2} \notin k$.

The proof is similar to that in Case 2 or Case 3; thus the detailed proof is omitted. In case $\text{char } k = 0$, we may apply Plans’s Theorem, i.e. Theorem 1.3. \square

§5. Other double covers of S_n

In this section we consider the rationality problem of G_n which is a double cover of the symmetric group other than $\widehat{S_n}$ and $\widetilde{S_n}$.

There are four double covers of the symmetric group S_n when $n \geq 4$. The trivial case is the split group $S_n \times C_2$. The rationality problem of the group $S_n \times C_2$ is easy because we may apply Theorem 2.6. It remains to consider the non-split cases: They are $\widehat{S_n}$, $\widetilde{S_n}$, and the group G_n defined below.

Definition 5.1 For $n \geq 3$, consider the group G_n such that the short exact sequence $1 \rightarrow \{\pm 1\} \rightarrow G_n \xrightarrow{p} S_n \rightarrow 1$ is induced by the cup product $\varepsilon_n \cup \varepsilon_n \in H^2(S_n, \{\pm 1\})$ (see, for example, [Se, page 654]) where $\varepsilon_n : S_n \rightarrow \{\pm 1\}$ is the signed map, i.e. $\varepsilon_n(\sigma) = -1$ if and only if $\sigma \in S_n$ is an odd permutation. Note that the group G_n is denoted by $\overline{S_n}$ in [Pl2].

The group G_n can be constructed explicitly as follows. Let $1 \rightarrow \{\pm 1\} \rightarrow C_4 = \{\pm\sqrt{-1}, \pm 1\} \xrightarrow{p_0} \{\pm 1\} \rightarrow 1$ be the short exact sequence defined by $p_0(\sqrt{-1}) = -1$. The group G_n can be realized as the pull-back of the following diagram

$$\begin{array}{ccc} & S_n & \\ & \downarrow \varepsilon_n & \\ C_4 & \xrightarrow{p_0} & \{\pm 1\}. \end{array}$$

Explicitly, as a subgroup of $S_n \times C_4$, $G_n = \{(\sigma, (\sqrt{-1})^i) \in S_n \times C_4 : \epsilon_n(\sigma) = p_0((\sqrt{-1})^i)\} = (A_n \times \{\pm 1\}) \cup \{(\sigma, \pm\sqrt{-1}) \in S_n \times C_4 : \sigma \notin A_n\}$.

If k is a field with $\text{char } k \neq 2$, a faithful $2n$ -dimensional representation can be defined as follows. Let $X = (\oplus_{1 \leq i \leq n} k \cdot x_i) \oplus (\oplus_{1 \leq i \leq n} k \cdot y_i)$ and G_n acts on X by, for $1 \leq i \leq n$,

$$(5.1) \quad \begin{aligned} t &: x_i \mapsto -x_i, \quad y_i \mapsto -y_i, \\ \tau &: x_i \mapsto x_{\tau(i)}, \quad y_i \mapsto y_{\sigma^{-1}\tau\sigma(i)} \\ \bar{\sigma} &: x_i \mapsto y_i \mapsto -x_i \end{aligned}$$

where $t = (1, -1) \in G_n \subset S_n \times C_4$, $\tau \in A_n$ and τ is identified with $(\tau, 1) \in G_n$, $\sigma = (1, 2) \in S_n$ and $\bar{\sigma} = (\sigma, \sqrt{-1}) \in G_n$.

The following theorem was proved by Plans [Pl2, Theorem 14 (b)] under the assumptions that $\text{char } k = 0$ and $\sqrt{-1} \in k$. Our proof is different from Plans's proof even in the situation $\text{char } k = 0$.

Theorem 5.2 *Assume that k is a field satisfying that (i) either $\text{char } k = 0$ or $\text{char } k = p > 0$ with $p \nmid n!$, and (ii) $\sqrt{-1} \in k$. Then $k(G_n)$ is k -rational for $n \geq 3$.*

Proof. Step 1. Apply Theorem 2.2. We find that $k(G_n)$ is rational over $k(x_i, y_i : 1 \leq i \leq n)^{G_n}$ where G_n acts on the rational function field $k(x_i, y_i : 1 \leq i \leq n)$ by Formula (5.1).

Step 2. Define $u_0 = \sum_{1 \leq i \leq n} x_i$, $v_0 = \sum_{1 \leq i \leq n} y_i$ and $u_i = x_i/u_0$, $v_i = y_i/v_0$ for $1 \leq i \leq n$. Note that $k(x_i, y_i : 1 \leq i \leq n) = k(u_j, v_j : 0 \leq j \leq n)$ with the relations $\sum_{1 \leq i \leq n} u_i = \sum_{1 \leq i \leq n} v_i = 1$. The action of G_n is given by

$$\begin{aligned} t &: u_0 \mapsto -u_0, \quad v_0 \mapsto -v_0, \quad u_i \mapsto u_i, \quad v_i \mapsto v_i, \\ \tau &: u_0 \mapsto u_0, \quad v_0 \mapsto v_0, \quad u_i \mapsto u_{\tau(i)}, \quad v_i \mapsto v_{\sigma^{-1}\tau\sigma(i)}, \\ \bar{\sigma} &: u_0 \mapsto v_0 \mapsto -v_0, \quad u_i \mapsto v_i \mapsto u_i \end{aligned}$$

where $1 \leq i \leq n$ and $t, \tau, \bar{\sigma}$ are defined in Formula (5.1).

Define $w_1 = u_0 v_0$, $w_2 = u_0/v_0$. Then $k(u_j, v_j : 0 \leq j \leq n)^{\langle t \rangle} = k(u_i, v_i : 1 \leq i \leq n)(w_1, w_2)$.

Note that $\tau(w_i) = w_i$ for $1 \leq i \leq 2$, $\bar{\sigma}(w_1) = -w_1$, $\bar{\sigma}(w_2) = -1/w_2$. By Theorem 2.1, $k(u_i, v_i : 1 \leq i \leq n)(w_1, w_2)^{G_n/\langle t \rangle} = k(u_i, v_i : 1 \leq i \leq n)(w_2)^{G_n/\langle t \rangle}(w')$ for some w' fixed by the action of $G_n/\langle t \rangle$. Moreover, we may identify $G_n/\langle t \rangle$ with S_n and identify $\bar{\sigma}$ (modulo $\langle t \rangle$) with σ .

Define $U_i = u_i - (1/n)$, $V_i = v_i - (1/n)$ for $1 \leq i \leq n$. We find that $\sum_{1 \leq i \leq n} U_i = \sum_{1 \leq i \leq n} V_i = 0$ and the action of S_n on $k(U_i, V_i : 1 \leq i \leq n)$ becomes linear. We will consider $k(U_i, V_i : 1 \leq i \leq n)(w_2)^{S_n}$. The action of S_n is given by

$$(5.2) \quad \begin{aligned} \tau &: U_i \mapsto U_{\tau(i)}, \quad V_i \mapsto V_{\sigma^{-1}\tau\sigma(i)}, \quad w_2 \mapsto w_2 \\ \sigma &: U_i \mapsto V_i \mapsto U_i, \quad w_2 \mapsto -1/w_2 \end{aligned}$$

where $1 \leq i \leq n$, $\tau \in A_n$, $\sigma = (1, 2)$ and $\sum_{1 \leq i \leq n} U_i = \sum_{1 \leq i \leq n} V_i = 0$.

Step 3. When $\text{char } k = 0$, there is a short-cut to prove this theorem. The proof of the general case when $\text{char } k \nmid n!$ will be postponed till Step 5.

Consider the action of S_n on the linear space $\sum_{1 \leq i \leq n} k \cdot U_i \oplus \sum_{1 \leq i \leq n} k \cdot V_i$. We will prove that the representation of S_n associated to this linear space is reducible.

Let $W = \sum_{1 \leq i \leq n} k \cdot s_i$ be the standard representation of S_n , i.e. $\sum_{1 \leq i \leq n} s_i = 0$ and $\lambda(s_i) = s_{\lambda(i)}$ for all $\lambda \in S_n$, for all $1 \leq i \leq n$. Let W' be the representation space of the tensor product of the standard representation and the linear character $\varepsilon_n : S_n \rightarrow \{\pm 1\}$. We will show that the representation associated to $\sum_{1 \leq i \leq n} k \cdot U_i \oplus \sum_{1 \leq i \leq n} k \cdot V_i$ is equivalent to that of $W \oplus W'$.

Since $\text{char } k = 0$, it suffices to show that the characters of these two representations are completely the same. This fact is easy to check for even permutations of S_n . As to the odd permutations, note that every odd permutation can be written as $\sigma\tau$ for some $\tau \in A_n$. Since $\sigma\tau(U_i) = V_{\tau(i)}$, $\sigma\tau(V_i) = U_{\sigma^{-1}\tau\sigma(i)}$, we find that the value of the character of $\sigma\tau$ for the representation associated to $\sum_{1 \leq i \leq n} k \cdot U_i \oplus \sum_{1 \leq i \leq n} k \cdot V_i$ is zero. Hence the result.

Step 4. In this paragraph we consider the general case when $\text{char } k \nmid n!$. Since $\sqrt{-1} \in k$, define $w = (\sqrt{-1} - w_2)/(\sqrt{-1} + w_2)$. We find that $\tau(w) = w$ for $\tau \in A_n$ and $\sigma(w) = -w$. Apply Theorem 2.1. We find that $k(u_i, v_i : 1 \leq i \leq n)(w_2)^{S_n} = k(U_i, V_i : 1 \leq i \leq n)^{S_n}(w'')$ for some w'' fixed by the action of S_n .

In particular, when $\text{char } k = 0$, apply Theorem 2.1 to $k(U_i, V_i : 1 \leq i \leq n)^{S_n}$. We find that $k(U_i, V_i : 1 \leq i \leq n)^{S_n} = k(W \oplus W')^{S_n} = k(s_i : 1 \leq i \leq n-1)^{S_n}(t_1, \dots, t_{n-1})$ where each t_i is fixed by S_n . Obviously the field $k(s_i : 1 \leq i \leq n-1)^{S_n}$ is k -rational. Hence the result.

Step 5. Now return to the general case when $\text{char } k \nmid n!$. Because of the above step, it suffices to show that $k(U_i, V_i : 1 \leq i \leq n)^{S_n}(X_1, \dots, X_N)$ is k -rational where $N = 2 \cdot (n!) - 2(n-1)$. Once we know $k(U_i, V_i : 1 \leq i \leq n)^{S_n}(X_1, \dots, X_N)$ is k -rational, we find that $k(G_n)$ is k -rational.

Since $\text{char } k \nmid n!$, we may embed the space $\sum_{1 \leq i \leq n} k \cdot U_i \oplus \sum_{1 \leq i \leq n} k \cdot V_i$ into the regular representation space $\oplus_{g \in S_n} k \cdot x(g)$. Thus Theorem 2.2 is applicable. We find that $k(x(g) : g \in S_n)^{S_n}$ is rational over $k(U_i, V_i : 1 \leq i \leq n)^{S_n}$. Explicitly $k(x(g) : g \in S_n)^{S_n} = k(U_i, V_i : 1 \leq i \leq n)^{S_n}(Y_1, \dots, Y_{N'})$ where $N' = n! - 2(n-1)$.

On the other hand, the regular representation space $\oplus_{g \in S_n} k \cdot x(g)$ contains the ordinary permutation action $\oplus_{1 \leq i \leq n} k \cdot z_i$ where S_n acts on z_1, \dots, z_n by $g \cdot z_i = z_{g(i)}$ for $1 \leq i \leq n$, $g \in S_n$. Apply Theorem 2.2 again. We find that $k(x(g) : g \in S_n)^{S_n}$ is rational over $k(z_i : 1 \leq i \leq n)^{S_n}$. Since $k(z_i : 1 \leq i \leq n)^{S_n}$ is k -rational, we find that $k(x(g) : g \in S_n)^{S_n}$ is k -rational. Hence $k(U_i, V_i : 1 \leq i \leq n)^{S_n}(X_1, \dots, X_N)$ is k -rational. Thus $k(G_n)$ is k -rational. \square

The first part of the following theorem was proved by Plans [Pl2, Theorem 14, (b)]; there he assumed that $\text{char } k = 0$.

Theorem 5.3 (1) If k is a field with $\text{char } k \neq 2$ or 3 , then $k(G_3)$ is k -rational.
(2) If k is a field with $\text{char } k \neq 2$ or 3 , then $k(G_4)$ is k -rational. Moreover, if $\text{char } k = 0$, then $k(G_5)$ is also k -rational.

Proof. Case 1. $n = 3$.

By Step 2 in the proof of Theorem 5.2 (note that the assumption $\sqrt{-1} \in k$ is used only till Step 4 there), it suffices to consider $k(U_i, V_i : 1 \leq i \leq 3)(w_2)^{S_3}$ where $\sum_{1 \leq i \leq 3} U_i = \sum_{1 \leq i \leq 3} V_i = 0$. Define $\tau = (1, 2, 3) \in S_3$. The actions are given as below,

$$\begin{aligned}\tau : U_1 &\mapsto U_2 \mapsto -U_1 - U_2, \quad V_2 \mapsto V_1 \mapsto -V_1 - V_2, \\ \sigma : U_1 &\leftrightarrow V_1, \quad U_2 \leftrightarrow V_2.\end{aligned}$$

Define $w_3 = U_1/V_2$, $w_4 = U_2/V_1$, $w_5 = V_1/V_2$. It follows that $k(U_i, V_i : 1 \leq i \leq 3)(w_2)^{S_3} = k(w_j : 2 \leq j \leq 5)(V_1)^{S_3} = k(w_j : 2 \leq j \leq 5)^{S_3}(w_0)$ for some w_0 by Theorem 2.1.

It remains to show that $k(w_j : 2 \leq j \leq 5)^{S_3}$ is k -rational. Note that

$$\begin{aligned}\tau : w_2 &\mapsto w_2, \quad w_3 \mapsto w_4 \mapsto (w_3 + w_4 w_5)/(1 + w_5). \\ \sigma : w_2 &\mapsto -1/w_2, \quad w_3 \mapsto 1/w_4, \quad w_4 \mapsto 1/w_3, \quad w_5 \mapsto w_3/(w_4 w_5).\end{aligned}$$

Define $w_6 = (w_3 + w_4 w_5)/(1 + w_5)$. Note that $k(w_3, w_4, w_5) = k(w_3, w_4, w_6)$ and

$$\begin{aligned}\tau : w_3 &\mapsto w_4 \mapsto w_6 \mapsto w_3, \\ \sigma : w_6 &\mapsto 1/w_6.\end{aligned}$$

Define $w_7 = (1 - w_3)/(1 + w_3)$, $w_8 = (1 - w_4)/(1 + w_4)$, $w_9 = (1 - w_6)/(1 + w_6)$. Then $k(w_3, w_4, w_6) = k(w_7, w_8, w_9)$ and $\tau : w_7 \mapsto w_8 \mapsto w_9 \mapsto w_7$, $\sigma : w_7 \mapsto -w_8$, $w_8 \mapsto -w_7$, $w_9 \mapsto -w_9$.

By Theorem 2.4 we find that $k(w_2, w_3, w_4, w_5)^{\langle \tau \rangle} = k(w_2, X_1, X_2, X_3)$ where $X_1 = w_7 + w_8 + w_9$ and

$$\begin{aligned}X_2 &= \frac{w_7^2 w_8 + w_8^2 w_9 + w_9^2 w_7 - 3w_7 w_8 w_9}{w_7^2 + w_8^2 + w_9^2 - w_7 w_8 - w_7 w_9 - w_8 w_9}, \\ X_3 &= \frac{w_7 w_8^2 + w_8 w_9^2 + w_9 w_7^2 - 3w_7 w_8 w_9}{w_7^2 + w_8^2 + w_9^2 - w_7 w_8 - w_7 w_9 - w_8 w_9}.\end{aligned}$$

Moreover, the action of σ is given by

$$\sigma : w_2 \mapsto -1/w_2, \quad X_1 \mapsto -X_1, \quad X_2 \mapsto -X_3, \quad X_3 \mapsto -X_2.$$

Apply Theorem 2.2. We find that $k(w_2, X_1, X_2, X_3)^{\langle \sigma \rangle} = k(w_2)^{\langle \sigma \rangle}(Y_1, Y_2, Y_3)$ for some Y_1, Y_2, Y_3 fixed by σ . Since $k(w_2)^{\langle \sigma \rangle}$ is k -rational, it follows that $k(w_2, X_1, X_2, X_3)^{\langle \sigma \rangle}$ is k -rational.

Case 2. $n = 4$.

Once again we use Step 2 in the proof of Theorem 5.2. It suffices to consider $k(U_i, V_i : 1 \leq i \leq 4)(w_2)^{S_4}$ where $\sum_{1 \leq i \leq 4} U_i = \sum_{1 \leq i \leq 4} V_i = 0$. Denote by $\lambda_1 = (1, 2)(3, 4)$, $\lambda_2 = (1, 3)(2, 4)$, $\rho = (1, 2, 3)$ and $\sigma = (1, 2)$ as before. Then S_4 is generated by $\lambda_1, \lambda_2, \rho, \sigma$.

Define $t_1 = U_1 + U_2, t_2 = V_1 + V_2, t_3 = U_1 + U_3, t_4 = V_2 + V_3, t_5 = U_2 + U_3, t_6 = V_1 + V_3$. The action of S_4 is given as follows,

$$\begin{aligned}\lambda_1 : t_1 &\mapsto t_1, t_2 \mapsto t_2, t_3 \mapsto -t_3, t_4 \mapsto -t_4, t_5 \mapsto -t_5, t_6 \mapsto -t_6, \\ \lambda_2 : t_1 &\mapsto -t_1, t_2 \mapsto -t_2, t_3 \mapsto t_3, t_4 \mapsto t_4, t_5 \mapsto -t_5, t_6 \mapsto -t_6, \\ \rho : t_1 &\mapsto t_5 \mapsto t_3 \mapsto t_1, t_2 \mapsto t_6 \mapsto t_4 \mapsto t_2, \\ \sigma : t_1 &\leftrightarrow t_2, t_3 \leftrightarrow t_6, t_4 \leftrightarrow t_5.\end{aligned}$$

It follows that $k(t_i : 1 \leq i \leq 6)(w_2)^{<\lambda_1, \lambda_2>} = k(T_i : 1 \leq i \leq 6)(w_2)$ where $T_1 = t_1/t_2, T_2 = t_3/t_4, T_3 = t_5/t_6, T_4 = t_2t_6/t_4, T_5 = t_4t_6/t_2, T_6 = t_2t_4/t_6$. Moreover, the actions of ρ and σ are given as,

$$\begin{aligned}\rho : T_1 &\mapsto T_3 \mapsto T_2 \mapsto T_1, T_4 \mapsto T_5 \mapsto T_6 \mapsto T_4, \\ \sigma : T_1 &\mapsto 1/T_1, T_2 \mapsto 1/T_3, T_3 \mapsto 1/T_2, \\ T_4 &\mapsto (T_1T_2/T_3)T_6, T_5 \mapsto (T_2T_3/T_1)T_5, T_6 \mapsto (T_1T_3/T_2)T_4.\end{aligned}$$

By Theorem 2.2, it suffices to show that $k(T_i : 1 \leq i \leq 3)(w_2)^{<\rho, \sigma>}$ is k -rational.

Define $w_3 = (1 - T_1)/(1 + T_1), w_4 = (1 - T_2)/(1 + T_2), w_5 = (1 - T_3)/(1 + T_3)$. Then we find

$$\begin{aligned}\rho : w_2 &\mapsto w_2, w_3 \mapsto w_5 \mapsto w_4 \mapsto w_3, \\ \sigma : w_2 &\mapsto -1/w_2, w_3 \mapsto -w_3, w_4 \mapsto -w_5, w_5 \mapsto -w_4.\end{aligned}$$

Use Theorem 2.4 to find $k(T_i : 1 \leq i \leq 3)(w_2)^{<\rho>}$. The remaining part of the proof is very similar to the last part of Case 1. The details are omitted.

Case 3. $n = 5$.

By [Pl2, Theorem 11], $k(G_5)$ is rational over $k(G_4)$. Since $k(G_4)$ is k -rational by Case 2, we are done. □

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